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LETTER TO THE EDITOR

# Universal amplitude of the free energy density in finite-size scaling: the Potts universality

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**Abstract.** Using the numerical results of the finite-size scaling study of the $q$-state Potts model by Blöte and Nightingale, we get the following conjectured expression for the universal amplitude of the free energy density:

$$A_0(u)=\pi\frac{(2-3u)(u+1)}{6(2-u)}$$

where

$$0\leqslant u\equiv\frac{2}{\pi}\cos^{-1}\left(\frac{\sqrt{q}}{2}\right)\leqslant 1.$$

According to Privman and Fisher (1984), the singular part of the free energy density on a cylinder-shaped system with size $V=L^{d-1}\times\infty$ near the critical point $t=0$, $h=0$ ($t=(T-T_c)/T_c$, $h=H/k_BT$) may be written as

$$f^{(s)}(t,h,L)=-\frac{F^{(s)}}{Vk_BT}\simeq L^{-d}Y(x_1,x_2)\qquad x_1=C_1tL^{y_t}\qquad x_2=C_2hL^{y_h}\tag{1}$$

where $Y(x,y)$ is a universal function, $y_t$ and $y_h$ are the thermal and magnetic exponents. $C_1$ and $C_2$ are non-universal metric factors.

In two dimensions, on a $L\times\infty$ strip with periodic boundary conditions built up of $L\times 1$ slices, the free energy density is given by

$$f_0(t,h,L)=(1/lL)\ln\Lambda_0(t,h,L)\tag{2}$$

where $\Lambda_0$ is the largest eigenvalue of the transfer matrix. Free energy levels $f_j(t,h,L)$ may be defined using the subdominant eigenvalues $\Lambda_j$ of the transfer matrix ($\Lambda_0>\Lambda_1\geqslant\Lambda_2\ldots$) through

$$f_j(t,h,L)=(1/lL)\ln\Lambda_j(t,h,L).\tag{3}$$

The singular part of the free energy levels is expected to behave as in equation (1):

$$f_j^{(s)}(t,h,L)\simeq L^{-2}Y_j(x_1,x_2).\tag{4}$$

The singular part is defined as

$$f_j(t,h,L)=f_j^{(s)}(t,h,L)+f_\infty(t,h)\tag{5}$$

where the last term is the analytic background which is the same for all the levels.

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At the critical point one gets

$$f_j = A_j L^{-2} + B \tag{6}$$

where $A_j = Y_j(0, 0)$ is a universal amplitude and $B$ is the critical value of the free energy density in the infinite system.

The correlation lengths are given by

$$\xi_{\parallel j}(t, h, L) = l\left[\ln\left(\frac{\Lambda_0}{\Lambda_j}\right)\right]^{-1} \tag{7}$$

($j = 1$: spin-spin correlations, $j = 2$: energy-energy correlations) so that, using equations (3) and (6) at the critical point, one gets

$$\xi_{\parallel j}(0, 0, L) = S_j L \tag{8}$$

where

$$S_j = (A_0 - A_j)^{-1}. \tag{9}$$

These correlation length universal amplitudes $S_j$ are known to be related to the decay exponents $\eta_j$ (Pichard and Sarma 1981, Luck 1982, Derrida and de Sèze 1982, Nightingale and Blöte 1983, Cardy 1984) through

$$S_j = 1/\pi\eta_j \tag{10}$$

so that

$$A_0 - A_j = \pi\eta_j. \tag{11}$$

In this letter, we deduce the universal amplitude of the free energy density $A_0$ from the results of a finite-size scaling study (Blöte and Nightingale 1982). A simple analytic expression is conjectured which is in excellent agreement with the numerical results and reproduces the known exact value $A_0 = \pi/12$ in the Ising case, $q = 2$ (Ferdinand and Fisher 1969).

Using the exact values for $B$ (Baxter 1973) and $f_0(L)$ for the three largest strips shown in table 1, the universal amplitudes $A_0(L)$ are obtained. A three-point fit is

**Table 1.** Free energy density at the critical point in the $q$-state Potts model as a function of the strip width $L$ (data taken from Blöte and Nightingale 1982). The values converge towards the exact result $B$ (Baxter 1973) for the infinite system.

| | $L$ | | | |
|---|---|---|---|---|
| $q$ | 9 | 10 | 11 | $\infty$ |
| 1/64 | −0.893 297 995 296 | −0.891 242 331 139 | −0.889 723 754 951 | −0.882 519 177 979 |
| 1/16 | −0.168 550 388 717 | −0.166 874 332 373 | −0.165 636 316 196 | −0.159 764 272 049 |
| 1/2 | 0.977 625 265 191 | 0.978 178 682 825 | 0.978 587 322 901 | 0.980 523 980 355 |
| 0.95 | 1.355 439 724 384 | 1.355 483 452 108 | 1.355 515 734 412 | 1.355 668 662 600 |
| 1.05 | 1.415 741 452 203 | 1.415 699 494 189 | 1.415 668 519 467 | 1.415 521 797 633 |
| 3 | 2.075 404 683 689 | 2.074 406 246 134 | 2.073 669 695 186 | 2.070 187 162 577 |
| 4 | 2.266 084 394 071 | 2.264 826 740 710 | 2.263 899 533 697 | 2.259 524 751 387 |

**Table 2.** Universal amplitude $A_{0\,\text{extrapolated}}$ obtained through a three-point fit of $A_0(L)$ using equation (12).

| | $A_0(L)$ | | | $A_{0\,\text{extrapolated}}$ | | |
|---|---|---|---|---|---|---|
| | $L$ | | | | | |
| $q$ | 9 | 10 | 11 | $\infty$ | $y$ | $C$ |
| 1/64 | −0.873 084 202 677 | −0.872 315 316 000 | −0.871 753 813 612 | −0.869 26 | 2.13 | −0.41 |
| 1/16 | −0.711 675 450 108 | −0.711 006 032 400 | −0.710 517 341 787 | −0.708 35 | 2.13 | −0.36 |
| 1/2 | −0.234 795 928 284 | −0.234 529 753 000 | −0.234 335 551 934 | −0.233 47 | 2.14 | −0.14 |
| 0.95 | −0.018 543 995 496 | −0.018 521 049 200 | −0.018 504 310 748 | −0.018 43 | 2.14 | −0.01 |
| 1.05 | 0.017 792 020 170 | 0.017 769 655 600 | 0.017 753 341 914 | 0.017 68 | 2.14 | 0.01 |
| 3 | 0.422 619 210 072 | 0.421 908 355 700 | 0.421 386 445 689 | 0.419 00 | 2.08 | 0.35 |
| 4 | 0.531 331 057 404 | 0.530 198 932 300 | 0.529 348 659 510 | 0.524 94 | 1.85 | 0.37 |

used to estimate $A_0$, assuming a power law correction to scaling

$$A_0(L) = A_0 + CL^{-y}. \tag{12}$$

The results are shown in table 2.

The exponents of the $q$-state Potts model are exactly known (den Nijs 1979, Black and Emery 1981, Nienhuis *et al* 1980, Pearson 1980, Wu 1982). Using the variable $u$ defined as

$$0 \leq u \equiv (2/\pi)\cos^{-1}(\sqrt{q}/2) \leq 1 \tag{13}$$

one has

$$\begin{aligned} y_t &= 3(1-u)/(2-u) \qquad q \leq 4 \\ y_h &= (3-u)(5-u)/4(2-u) \end{aligned} \tag{14}$$

**Table 3.** Universal amplitudes $A_0$, $A_1$ and $A_2$ for the singular part of the free energy levels of the $q$-state Potts model obtained from equations (16) and (17) in the text. $A_{0\,\text{extrapolated}}$ is presented for comparison.

| $q$ | $u$ | $A_{0\,\text{extrapolated}}$ | $A_0$ | $A_1$ | $A_2$ |
|---|---|---|---|---|---|
| 0 | 1 | | $-\pi/3 =$ −1.047 197 551 | $-\pi/3 =$ −1.047 197 551 | $-13\pi/3 =$ −13.613 568 17 |
| 1/64 | 0.960 185 314 | −0.869 26 | −0.869 154 052 | −0.987 051 486 | −12.713 771 87 |
| 1/16 | 0.920 213 825 | −0.708 35 | −0.708 256 312 | −0.931 130 284 | −11.881 819 72 |
| 1/2 | 0.769 946 544 | −0.233 47 | −0.233 438 107 | −0.753 415 910 | −9.274 429 021 |
| 0.95 | 0.675 934 685 | −0.018 43 | −0.018 426 991 | −0.662 744 760 | −7.971 364 109 |
| 1 | 2/3 | | 0 | $-5\pi/24 =$ −0.654 498 470 | $-5\pi/2 =$ −7.853 981 635 |
| 1.05 | 0.657 551 944 | 0.017 68 | 0.017 677 994 | −0.646 499 628 | −7.740 317 537 |
| 2 | 1/2 | | $\pi/12 =$ 0.261 799 388 | $-\pi/6 =$ −0.523 598 776 | $-23\pi/12 =$ −6.021 385 920 |
| 3 | 1/3 | 0.419 00 | $2\pi/15 =$ 0.418 879 021 | $-2\pi/15 =$ −0.418 879 021 | $-22\pi/15 =$ −4.607 669 226 |
| 4 | 0 | 0.524 94 | $\pi/6 =$ 0.523 598 776 | $-\pi/12 =$ −0.261 799 388 | $-5\pi/6 =$ −2.617 993 878 |

and the decay exponents for the spin-spin and energy-energy correlation functions are

$$\eta = \frac{1-u^2}{2(2-u)}$$
$$\eta_{\mathrm{EE}} = \frac{2(1+u)}{(2-u)}. \tag{15}$$

The numerical results obtained for $A_0$ may be fitted using the following expression (table 3):

$$A_0 = \pi \frac{(2-3u)(u+1)}{6(2-u)}. \tag{16}$$

The universal amplitudes for the second and third levels follow from equations (11) and (15):

$$A_1 = -\pi \frac{(1+u)}{6(2-u)}$$
$$A_2 = -\pi \frac{(1+u)(10+3u)}{6(2-u)}. \tag{17}$$

*Note added.* After this letter was submitted for publication, we learnt that the result given in equation (16) had been independently conjectured and derived from conformal invariance by Blöte *et al* (1986) (see also Affleck 1986).

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